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Farsightedly Stable Matchings*

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Abstract

We study the properties of von Neumann-Morgenstern farsightedly stable sets in application to matching models. We show that the result by Diamantoudi and Xue (2003) for hedonic games can be extended to a general matching with contracts framework: a collection of singleton stable sets constitutes a weak core of the matching with contracts game. We also show that singleton stable sets are invariant under different contractual languages.

1 Introduction

The notion of the abstract core is traditionally used to define the solution in matching problems. As Jordan (2006) notices the stability of an element of the core does not require any social construction or norm—this element is immune to any deviation. Of course, if we allow for a rich variety of deviations we usually run into a problem of an empty core. This problem is usually handled by assuming away some types of deviations.

In this paper we present an alternative approach. Instead of exogenous exclusion of certain types of deviations, let us allow the model to pick the ones that are irrelevant for stability. Some deviations will be labeled as *non-credible* by the solution concept.

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This solution concept coincides with a core endowed with a certain blocking relation. By looking at this blocking relation we can understand which deviations can be dropped from a problem. A social construction, or a social norm, will endogenously select the deviation types that are relevant. To achieve this, we use the farsighted version of von Neumann-Morgenstern stable sets as a solution for a matching problem.

We show that the collection of farsighted von Neumann-Morgenstern singleton stable sets constitutes a weak core under very mild assumptions on preference profiles' and matching structures. In particular it is enough to assume that agents only form preferences over their own contracts. No assumptions on the structure of allowed matchings, such as two-sidedness or supply chain structure (Ostrovsky, 2008), is required for the result.

We also show that farsighted von Neumann-Morgenstern singleton stable sets are insensitive to different formulation of contracts. In particular, it is not important whether the model is unitary or not (Kominers, 2012). This insensitivity stems from the fact that unwanted contracts are not rejected not because they are bundled with some other contracts but because they enable agents to access other desirable contracts.

Mauleon et al. (2011) show, that in the classic model of one-to-one matching by Gale and Shapley (1962) each element of the core is a von Neumann-Morgenstern Farsightedly stable set. They also extend this result to the case of many-to-one markets under the substitutable preferences.

2 Many-to-One Markets

In the current section, in contrast to Mauleon et al. (2011), we are going to drop the assumption of the substitutability and study the properties of von Neumann-Morgenstern farsightedly stable sets for general preference profiles.

We start with a model of many-to-one matchings with contracts. The set of agents is denoted by N and the set of available contracts is denoted by Ω. Each contract $\omega \in \Omega$ specifies two participants $a(\omega) \in N^2$ and possibly some other features of this contract. We allow multiple contracts between two agents, i.e. $\omega_1, \omega_2 \in \Omega : \omega_1 \neq \omega_2$ and $a(\omega_1) = a(\omega_2)$.

Each agent $i \in N$ has preferences over subsets of contracts $\succeq_i \subset 2^\Omega \times 2^\Omega$. In this model agents, in general, agents not only care about their own contracts but also about contracts of all other agents. This framework is similar to one used in Hatfield et al. (2012).

In the current section we will impose several other assumptions on this model. We start with the assumption that there are two sides of the market. Following the tradition in the literature, we call them 'doctors' and 'hospitals': $D \cup H = N$ and $D \cap H = \emptyset$. Furthermore, each doctor can only be employed by one hospital. Under our assumption about the contracts, doctor's preferences are defined on the set of contracts available with hospitals. The preferences are strict. Each doctor (and hospital) has an outside option of staying without the match. We denote this outside option by $\emptyset$. For each $d \in D$, $\succ_d$ represents the the preferences of a doctor d.

Hospitals can hire more than one doctor. The preferences of a hospital are defined on the sets of contracts available. As in the case of doctors, the preferences of hospitals are strict. For each $h \in H$, $\succ_h$ represents the preferences of a hospital h.

Let $\gamma \subset \Omega$ be a matching. For convenience, by $\gamma(i,j)$ we denote the set of contracts between agents i and j in γ:

$$\gamma(i,j) = \{\omega \in \gamma \mid a(\omega) = \{i,j\}\}$$

In addition we define what matchings are individually rational.

Definition 1. *A matching μ is individually rational if for all $i \in N : \mu \succeq_i \emptyset$.*

In this definition we follow Konishi and Unver (2006), who make an important distinction between notions of individual stability and individual rationality. This distinction is crucial for our result in the next section: Traditional notions of stability require any stable matching to be individually stable, i.e. require each agent not to keep contracts that he does not like among his stable contracts. We show that sometimes agents may want to go for some contracts that they dislike in order to access contracts that they desire.

2.1 Example 1: Stability without substitutable preferences

Hatfield and Milgrom (2005) show that if hospitals' preferences satisfy substitutability condition, stable set of contracts always exist[1]. However, stability is not guaranteed if the substitutability assumption is dropped (Hatfield and Kojima (2010), introduce a new assumption, called bilateral substitutability, that relaxes the substitutability, but still guarantees the existence of stable matchings). The following example suggests, that in the absence of the substitutability, stability may be

[1]Hatfield and Milgrom (2005) assume that a doctor cannot sign more than one contract, so no further restriction on doctors' preferences is required for this result.

too demanding, and there may exist a reasonable alternative, that is less sensitive to the properties of preferences.

Imagine a market that consists of two hospitals, h_1, and h_2, and two doctors, d_1, and d_2. The preferences are:

$$\begin{array}{llcll} h_1: & \{d_1 d_2\} \succ \emptyset & \qquad & d_1: & h_1 \succ h_2 \succ \emptyset \\ h_2: & d_1 \succ \{d_1 d_2\} \succ d_2 \succ \emptyset & & d_2: & h_2 \succ h_1 \succ \emptyset \end{array}$$

Observe, that preferences of hospital h_1 are not substitutable. Also, it is easy to see, that there exist no stable matching in this example. Indeed, the matching in which both doctors are employed by hospital h_1 is blocked by pair h_2, d_2, the matching in which doctor d_1 is employed by hospital h_2 and doctor d_2 is unemployed is blocked by the coalition of h_1, d_1 and d_2 and finally the matching in which doctor d_1 is unemployed is blocked by pair h_2, d_1. All other matchings fail to satisfy individual stability.

Figure 1: Matchings μ_1 (on the left) and μ_2 (on the right).

Matching μ_1 (see fig. 1) is not stable because it does not satisfy individual stability property for hospital h_2: if h_2 has both doctors available for employment, the hospital prefers to hire doctor d_1 alone. Alternatively, if h_2 has ongoing contracts with both doctors, the hospital is better off if doctor d_2 is fired, since $d_1 \succ_{h_2} \{d_1 d_2\}$.

Note, however, that firing doctor d_2 is a short-sighted move. If d_2 is unemployed, hospital h_1 has a chance of hiring both doctors. Doctor d_2 prefers working in hospital h_1 to being unemployed and doctor d_1 prefers hospital h_1 to h_2. Therefore as soon as doctor d_2 is fired, hospital h_1 can make offers to both doctors and hospital h_2 will loose the contract with doctor d_1. The ultimate consequence of firing doctor d_2 for hospital h_2 is that it will be left with no employees, which is worse than having contracts with both doctors. This consideration should stop hospital h_2 from firing doctor d_2.

2.2 Farsighted von Neumann-Morgenstern stability

The example above suggests, that agents have to consider the long run consequences of their actions. This approach was originally suggested by Harsanyi (1974) in application to von Neumann and Morgenstern stable sets. It is often used to model forward looking agents in cooperative models (see for example Greenberg (1990), Chwe (1994), and Ray and Vohra (1997)). To incorporate this approach in a model of matching we have to redefine stability concept. Instead of the core-like stability we will use von Neumann-Morgenstern stable sets defined on farsighted blocking relations (as in Chwe (1994), Diamantoudi and Xue (2007), Mauleon et al. (2011) and many others).

It is convenient to think of a two-sided matching with as a bipartite graph. The nodes denote agents. Each edge is endowed with a description of a contract between agents on the two ends of the edge.

Here we tailor a particular formulation of farsighted von Neumann-Morgenstern stability used in Roketskiy (2012) to fit the current framework. Similar formulations were used in Diamantoudi and Xue (2007), Herings et al. (2009) and Mauleon et al. (2011).

Our solution concept relies on three components: Enforcability notion, blocking notion and abstract von Neumann-Morgenstern stable sets. We start with enforcability notion.

Definition 2. *A coalition S can enforce a transition from a matching ψ to γ or*

$$\psi \overset{S}{\to} \gamma$$

if for all $i, j \in N, i \neq j$ the following holds:

(i) $\gamma(i,j) \subsetneq \psi(i,j) \implies \{i,j\} \cap S \neq \emptyset$;

(ii) $\gamma(i,j) \setminus \psi(i,j) \neq \emptyset \implies \{i,j\} \subset S$.

In words, a new contract appears if both participants consent, and an existing contract can be seized unilaterally by any of the two participants. The enforcability notion can be paired with agents preferences to obtain a blocking notion:

Definition 3. *A matching ψ setwise farsightedly blocks γ or*

$$\psi \rhd \gamma$$

if there exists a sequence $\{(S_i, \phi_i)\}_{i=1}^{K}$, $\forall i = 1, ..., K : S_i \subset N$ and $\phi_i \subset \Omega$ such, that

(i) $\gamma = \phi_1 \overset{S_1}{\to} \phi_2 \overset{S_2}{\to} \dots \overset{S_K}{\to} \psi$

(ii) $\psi \succ_{S_k} \phi_k$ *for all* $k \le K$.

A sequence $\{(S_i, \phi_i)\}_{i=1}^{K}$ *that satisfies properties (i) and (ii) is called* γ*-to-*ψ *path.*

This notion operates on an implicit assumption that a network ψ will remain unaltered in the long run. If it is the case, every agent instrumental for a transition from γ to ψ will act voluntarily to carry out this transition. Finally, the implicit assumption mentioned above is reinforced in the following definition:

Definition 4. *A set of matchings* $\mathcal{R}$ *is von Neumann-Morgenstern farsightedly stable, if it satisfies the following two conditions:*

(IS) *for any* $\psi, \gamma \in \mathcal{R} : \psi \not\triangleright \gamma$;

(ES) *for any* $\gamma \notin \mathcal{R}$ *there exist* $\psi \in \mathcal{R} : \psi \triangleright \gamma$.

Put it different, agents consider a set of matchings stable if it consists of all matchings that are unblocked by stable ones.

2.3 Characterization of $\triangleright$

The following lemma characterizes $\triangleright$. It is our main tool in studying the properties of stable sets for many-to-one matching problems.

Lemma 1. *Let* μ *be an individually rational matching. Then,* $\mu \triangleright \nu$ *if and only if* $\nexists h \in H$ *such that* $\forall Z \subset \nu(h) : \mu \succ_Z \nu$,[2]

$$\nu(h) \backslash Z \succ_h \mu.$$

Proof. Assume, by contradiction, that $\mu \triangleright \nu$ and $\exists h \in H$ such, that $\forall Z \subset \nu(h) :$ $\mu \succ_Z \nu$,

$$\nu(h) \backslash Z \succ_h \mu.$$

Since $\nu \succ_h \mu$, it must be that $\nu(h) \neq \mu(h)$. By definition of $\triangleright$, there exists a sequence $\{(\mu_i, S_i)\}_{i=1}^{K}$, in which some set of doctors $Z \subset \nu(h) : \mu \succ_Z h$ breaks the contracts with hospital h. Notice, that $\nu(h) \backslash Z \succ_h \mu$, and therefore $\nu(h) \backslash Z \neq \mu(h)$. The sequence must stop in ν, hence the doctor h must participate in enforcing the matching μ at some moment in the sequence, hence it must be that $\mu \succ_h \nu(h) \backslash Z$ which is a contradiction.

[2] If $Z = \emptyset$, we set $\nu \succ_Z \mu$ to be true for any ν and μ.

To prove the reverse, we construct the sequence, that guarantees, that $\mu \rhd \nu$. Take a set of hospitals that are not indifferent between two matchings: $\widehat{H} = \{h \mid \nu(h) \neq \mu(h)\}$. Lets index the set $\widehat{H}$ in an arbitrary way. Observe, that for any hospital $h_k \in \widehat{H}$, there exist a set of doctors Z_k (possibly empty) such, that $\mu \succ_{Z_k} \nu$ and $\mu \succ_{h_k} \nu(h_k)\backslash Z_k$. Set $S_{2k-1} = Z_k$ and $S_{2k} = h_k$. Also, set the induction rule

$$\nu_{2k}(x) = \begin{cases} \emptyset, & \text{if } x \in S_{2k-1} \\ \nu_{2k-1}(x)\backslash S_{2k-1}, & \text{if } \mu(x) \supset S_{2k-1} \\ \nu_{2k-1}(x), & \text{otherwise,} \end{cases}$$

$$\nu_{2k+1}(x) = \begin{cases} \emptyset, & \text{if } x = S_{2k} \\ \emptyset, & \text{if } \mu(x) = S_{2k} \\ \nu_{2k}(x), & \text{otherwise.} \end{cases}$$

We start the induction at $\nu_1 = \nu$ and proceed until we run out of set $\widehat{H}$. The matching $\nu_{2|\widehat{H}|+1}$ is such, that all the hospitals from set $\widehat{H}$ have no doctors and all the hospitals from set $H\backslash\widehat{H}$ are matched as in μ. Finally we can set $S_{2|\widehat{H}|+1} = \widehat{H} \cup \{d \mid \nu_{2|\widehat{H}|+1}(d) = \emptyset\}$. The sequence ν_i can clearly be enforced by the sequence of coalition S_i. We only need to make sure, that agents' incentives are properly set up along the sequence. Indeed, at each odd step $2k-1$ except the last one, a coalition of doctors $S_{2k-1} = Z_k$ break their contracts with a hospital h_k and we know, that $\mu \succ_{Z_k} h_k$. At each even step $2k$, a doctor h_k gets rid of the remaining doctors $\nu(h_k)\backslash Z_k$, and by construction $\mu_{h_k} \succ_{h_k} \nu(h_k)\backslash Z_k$. Finally the last step goes through, since all agents in $S_{2|\widehat{H}|+1}$ have no contracts in $\nu_{2|\widehat{H}|+1}$ and μ is individually rational.

$\square$

Using this lemma we can replicate the result due to Diamantoudi and Xue (2003). Strictly speaking, the theorem of Diamantoudi and Xue (2003) does not apply to this situation. The case of many-to-one matching can be modeled as a hedonic coalition formation game, in which each coalition is a hospital and doctors hired by this hospital. Clearly, the coalitions are well defined because each doctor can only be assigned to one hospital. The induced preferences defined over the coalitions are now not strict, because in our model, each doctor only cares about the hospital he is assigned and does not care about his potential coworkers. Diamantoudi and Xue (2003) assume that the preferences over the coalitions are strict. However, we show that their logic can be easily extended to our model.

Proposition 1. *A matching μ is in a weak core if and only if $\{\mu\}$ is von Neumann-Morgenstern farsightedly stable.*

Proof. We first prove that stability implies von Neumann-Morgenstern farsighted stability. Take a stable matching μ. By definition of stability, μ is individually rational. By contradiction, assume that $\{\mu\}$ is not von Neumann-Morgenstern farsightedly stable. Then, there exist a matching ν such, that $\mu \not\triangleright \nu$. By lemma 1, there exist a hospital h such, that $\forall Z \subset \nu(h) : \mu \succ_Z \nu$,

$$\nu(h)\backslash Z \succ_h \mu.$$

Take a largest Z, that satisfies this property. Since μ is individually rational, $S = \nu(h)\backslash Z \neq \emptyset$. Also $h \succ_S \mu$. The coalition (h, S) blocks the matching μ, hence the contradiction.

To prove the reverse, we assume by contradiction that there exist a coalition (h, S), that blocks von Neumann-Morgenstern farsighted stable matching μ, i.e. $h \succ_S \mu$ and $S \succ_h \mu$. By lemma [IR], μ is individually rational, and hence lemma 1 is applicable. Construct a matching ν, in which the doctors in S are employed by the hospital h, i.e. $\nu(h) = S$, and the rest of the agents are matched in an arbitrary fashion. By assumption, it must be that $\mu \triangleright \nu$. By (the negation of) lemma 1, for any hospital x there exist a (possibly empty) set of doctors $Z \in \nu(x)$ such that $\mu \succ_Z x$ and $\mu \succ_x \nu(x)\backslash Z$. Since, $S = \nu(h) \succ_S \mu$, the only set Z that satisfies the property $\mu \succ_Z h$ is the empty set. This means, that $\mu \succ_h \nu(h) = S$, which is a desired contradiction, since (h, S) block μ. □

Let us go back to the example from section 2.1 and apply the above proposition. Even though, there is no stable matchings, the weak core of the game is non-empt: Matching μ_1 (see fig. 1) is unblocked by any autonomous coalition. Therefore, μ_1 is a singleton von Neumann-Morgenstern stable set.

2.4 Example 2: Nonsingleton stable set

Proposition 1 characterizes the collection of all singleton von Neumann-Morgenstern stable sets. In general stable sets do not have to be singletons. In this section we provide an example in which the set of stable matchings and a weak core are both empty but there exists a stable set that consists of three matchings. The following result states that the smallest possible nonsingleton stable set must be of size 3.

Proposition 2. *If $\mathcal{M}$ is von Neumann-Morgenstern farsightedly stable, then $|\mathcal{M}| \neq 2$.*

Proof. If $\mathcal{M}$ is singleton, the proposition is correct. Suppose that $|\mathcal{M}| > 1$. Take $\mu, \nu \in \mathcal{M}$. By the definition $\mu \not\succ \nu$. By lemma 1, there exist an h_ν such, that $\forall Z \subset \nu(h_\nu) : \mu \succ_Z \nu$,

$$\nu(h_\nu)\backslash Z \succ_{h_\nu} \mu.$$

. Analogously we have $\nu \not\succ \mu$, and hence there exists hospital h_μ defined the same way as h_ν. First, observe, that $h_\nu \neq h_\mu$, since $\nu \succ_{h_\nu} \mu$ and $\mu \succ_{h_\mu} \nu$. Let

$$\begin{aligned} S_\mu &= \mu(h_\mu)\backslash\{d \in \mu(h_\mu) \mid \nu \succ_d \mu\} \\ S_\nu &= \nu(h_\nu)\backslash\{d \in \nu(h_\nu) \mid \mu \succ_d \nu\} \end{aligned}$$

Observe, that $S_{h_\nu} \cap S_{h_\mu} = \emptyset$ (otherewise, there would exist a doctor d such that $\nu \succ_d \mu$ and $\mu \succ_d \nu$). We can construct a new matching ρ, such that $\rho(h_\nu) = S_\nu, \rho(h_\mu) = S_\mu$ and the rest of the agents are matched arbitrary. By lemma 1, $\mu \not\succ \rho$ and $\nu \not\succ \rho$, hence there exists the third element in $\mathcal{M}$.

□

The example we present in this section has a von Neumann-Morgenstern farsightedly stable set of size 3.

Imagine, that we have six hospitals and six doctors. The set of doctors is $D = \{d_1, d_2, d_3, b_1, b_2, b_3\}$ and the sef of hospitals is $H = \{h_{12}, h_{13}, h_{21}, h_{31}, h_{23}, h_{32}\}$[3].

The preferences of doctors and hospitals are the following:

$$\begin{array}{llcll} h_{12}: & \{d_1 d_3\} \succ b_1 \succ \emptyset & \quad & d_1: & h_{12} \succ h_{23} \succ h_{21} \succ \emptyset \\ h_{21}: & \{b_1 b_2\} \succ d_1 \succ \emptyset & & d_2: & h_{23} \succ h_{31} \succ h_{32} \succ \emptyset \\ h_{23}: & \{d_1 d_2\} \succ b_2 \succ \emptyset & & d_3: & h_{31} \succ h_{12} \succ h_{13} \succ \emptyset \\ h_{32}: & \{b_2 b_3\} \succ d_2 \succ \emptyset & & b_1: & h_{21} \succ h_{13} \succ h_{12} \succ \emptyset \\ h_{13}: & \{b_1 b_3\} \succ d_3 \succ \emptyset & & b_2: & h_{32} \succ h_{21} \succ h_{23} \succ \emptyset \\ h_{31}: & \{d_2 d_3\} \succ b_3 \succ \emptyset & & b_3: & h_{13} \succ h_{32} \succ h_{31} \succ \emptyset \end{array}$$

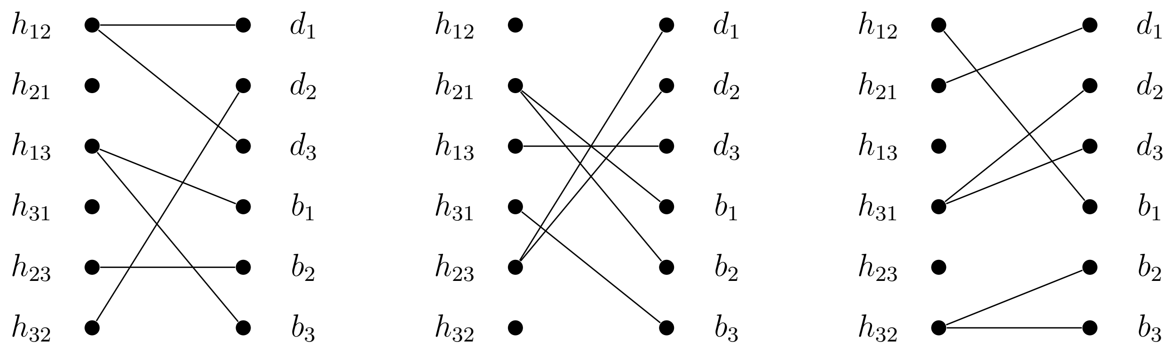
The three matchings μ_1, μ_2 and μ_3 presented on the fig. 2 form a stable set.

First, observe that this example is constructed in such a way that the weak core is empty. If in some individually stable matching a hospital has only one employee this employee can always do better if he finds a pair and joins some other hospital.

For example that the coalition $\{h_{31}, d_2, d_3\}$ blocks matching μ_1. Nonetheless we consider μ_1 to be stable. The reason is that the deviation by coalition $\{h_{31}, d_2, d_3\}$ is not credible: since d_1 is going to be on the market, the coalition $\{h_{23}, d_1, d_2\}$ can counterblock the deviation.

[3]An unusual indexing of doctors and hospitals is going to be clear, once more details of the example are presented.

Figure 2: Matchings μ_1 (on the left) and μ_2 (in the center) and μ_3 (on the right).



This example illustrates the important point about the existence of stable sets: It is not caused merely by relaxing individual stability to individual rationality (i.e. by switching from stability to weak core). Stable sets may exist even in models for which sets of stable matchings and weak core are both empty. It does not, however, mean that stable sets are guaranteed to exist. For instance, the simple three-person roommate problem does not have a farsighted von Neumann-Morgenstern stable set (see Example 1 in Klaus et al. (2011)).

3 Networks

In this section we present results for general networks. For this section we drop the assumption that matching is two-sided. This brings us to a setup similar to Hatfield et al. (2012).

3.1 Weak Core

Our first result shows that farsightedly von Neumann-Morgenstern stable singletons constitute a weak core. This theorem generalizes the result presented in Proposition 1.

Theorem 5. *Suppose that agents only care about their own contracts, i.e. for all* $i \in N$ *and for any two matchings* $\gamma_1, \gamma_2 : \gamma_1(i) = \gamma_2(i) \implies \gamma_1 \sim_i \gamma_2$. *Then, a singleton* $\{\mu\}$ *is farsightedly von Neumann-Morgenstern stable if and only if matching* μ *belongs to a weak core.*

Proof. Take a matching μ that is farsightedly von Neumann-Morgenstern stable and assume by contradiction that it does not belong to a weak core: $\exists \nu$ such that for $S = \{i \in N \mid \nu(i) \neq \emptyset\}$ we have $\nu \succ_S \mu$. This is a contradiction to $\mu \rhd \nu$.

Take a matching μ that is in a weak core and some other arbitrary matching ν. We aim to show that $\mu \rhd \nu$, i.e. there exists a sequence of matchings that starts with ν and ends with μ, such that each agent is better off at μ than at the intermediate step when he makes changes to his set of contracts. Set $\tau_0 = \nu$ and $\tau_\infty = \mu$. Since μ belongs to a weak core, there exists a set of agents $S_0 : \tau_\infty \succ_{S_0} \tau_0$. Let's look at a matching ν_1 obtained from matching ν_0 by dropping all contracts with agents in S_0. In this new matching we also can find agents who prefer μ over it. In order to proceed with the argument we setup the induction. Take a matching ν_k that contains at least one contract. There exists a set of agents S_k who have some contracts in ν_k and prefer μ over ν_k (otherwise the set of all agents with contracts would block μ, hence the contradiction). Obtain a matching ν_{k+1} by deleting all contracts of S_k from ν_k. Starting from ν_0 this induction will stop at the empty matching ν_K. It is easy to see that the sequence $(\nu_0, S_0), ..., (\nu_K, S_K), (\nu_\infty, N)$ supports $\mu \rhd \nu$. $\square$

Although weak core is usually not considered in the matching literature because it imposes very few explicit restriction on matching, we show that it is an interesting solution concept because it implicitly incorporates farsighted behavior when we restrict agents' preferences to be only over their own contracts.

3.2 Contractual Language

In this section we investigate the relationship between farsightedly von Neumann-Morgenstern stable sets and so called contractual languages. We show that singleton stable sets are invariant under different contractual languages as long as contractual primitives stay unchanged. Put it differently, it does not matter if the relationship between two agents is specified using one or multiple contracts as long as the content of this relationship (which in our model is defined by agents' preferences) is the same. This intuition follows through because the agents bargain over all their contracts at the same time. Therefore the agents can condition fulfilling their obligations under one contract on their counterpart fulfilling obligations under some other contract.

The issue of the distribution of contractual primitives between actual contracts and its relation to the stability concepts is raised in a recent paper by Hatfield and Kominers (2012). They introduce a notion of a contractual language and argue that the traditional solutions are sensitive to the way one specifies a contractual language.

Of course, in reality, it matters how content is distributed among different contracts. The reasons for this lies beyond the scope of this model: Actual *legal* con-

tracts specify not only the agents obligations but also what follows the breach of these contracts. Our model ignores the issue of enforcement of contracts and we implicitly assume that once the agents agree on a set of contracts, these contracts are binding. Instead of the enforcement issues we are mostly interested in pre-contract negotiations: We operate on the presumption that negotiations stop and contracts are signed as soon as parties arrive at the stable set of contracts' drafts.

In order to proceed we first introduce several useful notions into our model. Take a matching problem $(\Omega, N, \succ)$. Define a new set of contracts $\widehat{\Omega}$ and a correspondence $C : \widehat{\Omega} \to 2^{\Omega}$ such that for any contract $\omega \in \widehat{\Omega} : a(\omega) = a(C(\omega))$. This correspondence C defines bundling of certain sets of contracts from Ω into indivisible contracts in $\widehat{\Omega}$. The later condition makes sure that C only bundles contracts that belong to the same pairs of agents.

To define a new matching problem using bundled contracts in $\widehat{\Omega}$ we must define new preferences of agents that agree with $\succ$. Take an agent $i \in N$ and any two sets of contracts $\mu, \nu \subset \widehat{\Omega}$. We say that $\mu \succ_i^C \nu$ if $C(\mu) \succ_i C(\nu)$.

Theorem 6. *Take two matching problems $(\Omega, N, \succ)$ and $(\widehat{\Omega}, N, \succ^C)$, where C is a correspondence defined above. Let $\mathcal{R}$ be a collection of singleton von Neumann-Morgenstern farsightedly stable sets for $(\Omega, N, \succ)$.*

If for any $\rho \in \mathcal{R}$ there exists $\widehat{\rho} \subset \widehat{\Omega} : C(\widehat{\rho}) = \rho$, then $\widehat{\mathcal{R}} = \{\widehat{\rho} \subset \widehat{\Omega} \mid C(\widehat{\rho}) \in \mathcal{R}\}$ is a collection of singleton von Neumann-Morgenstern farsightedly stable sets for $(\widehat{\Omega}, N, \succ^C)$.

Proof. First we will show that for any sets of contracts $\mu, \nu : \mu \not\triangleright \nu$ and for any two sets of contracts $\widehat{\mu}, \widehat{\nu} : C(\widehat{\mu}) = \mu, C(\widehat{\nu}) = \nu$ we have $\widehat{\mu} \not\triangleright \widehat{\nu}$. By contradiction, assume that $\widehat{\mu} \not\triangleright \widehat{\nu}$. Take a sequence of transitions from $\widehat{\nu}$ to $\widehat{\mu}$ and apply C to each element of this sequence. The resulting sequence supports $\mu \triangleright \nu$ which is a contradiction.

This observation is sufficient to show that if $\widehat{\mu}$ is a singleton stable set for $(\widehat{\Omega}, N, \succ^C)$, then $\mu = C(\widehat{\mu})$ is a singleton stable set for $(\Omega, N, \succ)$. Note also that internal stability of nonsingleton sets trivially follows from this observation.

Take matching μ that is a singleton stable set for $(\Omega, N, \succ)$. We will show that $\widehat{\mu} : \mu = C(\widehat{\mu})$ is stable for $(\widehat{\Omega}, N, \succ^C)$. Take any $\widehat{\nu}$. Note, that $\mu \triangleright \nu = C(\widehat{\nu})$ hence there exists a ν-to-μ path. Convert the elements of this path in such a way that $\widehat{\nu}_i : \nu_i = C(\widehat{\nu}_i)$ wherever it is possible. Take a smallest i for which this conversion is impossible, and instead of taking ν_i, take $\nu_i^{-S_i} = \nu_i \setminus \{\omega \in \Omega \mid a(\omega) \cap S_i \neq \emptyset\}$. We claim that there exists $\widehat{\nu}_i : \nu_i^{-S_i} = C(\widehat{\nu}_i)$. Also, observe that $\mu \triangleright \nu_i^{-S_i}$, so we can now look at the $\nu_i^{-S_i}$-to-μ path and proceed by induction to construct $\widehat{\nu}$-to-$\widehat{\mu}$ path. $\square$

The proof of the proposition above cannot be trivially extended to the case of stable sets of arbitrary size. To see that, note that $\mu \triangleright \nu$ does not imply $\widehat{\mu} \not\triangleright \widehat{\nu}$. Here

is the example with 3 agents. The set of contracts is $\{a, b, c, d, e, f\}$ where contracts a and d are between agents 1 and 3 and the rest are between agents 2 and 3. The preferences are the following:

$$\begin{array}{ll} 1: & a \succ d \succ \emptyset \\ 2: & \{e, f\} \succ \{b, c\} \succ \emptyset \\ 3: & \{a, b, c\} \succ a \succ \{e, f, d\} \succ \emptyset \end{array}$$

Here we have $\{e, f, d\} \rhd \{a, b, c\}$. However, if contracts b and c are bundled into one indivisible contract $\widehat{b}$, we have $\{e, f, d\} \not\rhd \{a, \widehat{b}\}$. This happens because agent 2, who is the only one that prefers $\{e, f, d\}$ over $\{a, b, c\}$, can not influence agent 3 enough for the latter to drop a contract a. Nevertheless, both $\{a, b, c\}$ and $\{a, \widehat{b}\}$ are stable sets.

Given this observation, we conjecture that the result in Theorem 6 holds not only for singleton but also for any other stable sets.

The issue of the distribution of contractual primitives across contracts is related to a notion of unitarity of matching models. This notion is discussed in Kominers (2012) in detail.

4 Concluding remarks

In this paper we study properties of farsighted von Neumann-Morgenstern stable sets in application to matching problems. We show that singleton stable sets constitute the weak core of the game. We also show that stable sets feature a certain invariance with respect to different contractual languages.

Using some examples we show that von Neumann-Morgenstern stable sets may exist when agents preferences do not satisfy substitutability condition and as a result a set of stable matchings is empty. We conjecture that von Neumann-Morgenstern stable sets may generally exist under milder conditions than ones needed for the existence of stable matchings. The characterizations of such conditions is an open question for the future research.

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